

Modeling error in Approximate Deconvolution Models

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Abstract

We investigate the asymptotic behaviour of the modeling error in approximate deconvolution model in the 3D periodic case, when the order N of deconvolution goes to ∞ . We consider successively the generalised Helmholtz filters of order p and the Gaussian filter. For Helmholtz filters, we estimate the rate of convergence to zero thanks to energy budgets, Gronwall's Lemma and sharp inequalities about Fourier coefficients of the residual stress. We next show why the same analysis does not allow to conclude convergence to zero of the error modeling in the case of Gaussian filter, leaving open issues.

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1 Introduction

Direct Numerical Simulations of flows from the Navier-Stokes Equations (NSE)

$$(1.1) \quad \begin{aligned} \mathbf{u}_t + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}(\mathbf{x}, 0) &= \mathbf{u}_0(\mathbf{x}), \end{aligned}$$

are accurate only for small Reynold numbers. For large Reynolds numbers, flows are turbulent and only means or large scales of velocity and pressure fields might be computed thanks to turbulent models.

Large Eddy Simulation (LES) modeling of turbulent flows aims to apply to the NSE a low pass filter specified by a convolution kernel G , leading to the filtered NSE, written in the form

$$(1.2) \quad \begin{aligned} \bar{\mathbf{u}}_t + \nabla \cdot (\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) - \nu \Delta \bar{\mathbf{u}} + \nabla \bar{p} &= \bar{\mathbf{f}} + \nabla \cdot \mathbb{S}(\mathbf{u}, \mathbf{u}), \\ \nabla \cdot \bar{\mathbf{u}} &= 0, \\ \bar{\mathbf{u}}(\mathbf{x}, 0) &= \bar{\mathbf{u}}_0(\mathbf{x}), \end{aligned}$$

where $\bar{\mathbf{u}} = G \star \mathbf{u}$ is the large scale velocity, $\bar{p} = G \star p$ the large scale pressure,

$$(1.3) \quad \mathbb{S}(\mathbf{u}, \mathbf{u}) = \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} - \overline{\mathbf{u} \otimes \mathbf{u}},$$

is the subfilter scale stress tensor. A modelisation process aims to seek for suitable approximations to $\mathbb{S}(\mathbf{u}, \mathbf{u})$ in terms of $\bar{\mathbf{u}}$ to close System (1.2), that yields a LES model [3, 7, 16].

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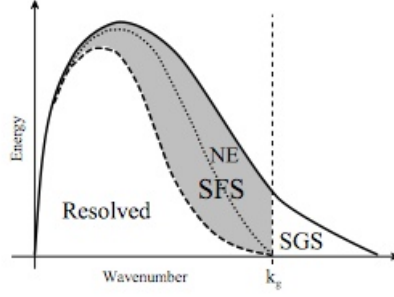


Figure 1: *From Chow et al. 2005 [5]. American Meteorological Society. Reprinted with permission.*

Most of LES models are over diffusive and trend to underestimate the energy, creating a subfilter scale region (SFS). The total error committed is the sum of the numerical error NE and the SFS area [5]. To reduce the SFS area, one uses to apply a deconvolution operator to the filter [5, 9, 19, 11, 12].

The aim of this paper is to estimate the error modeling in terms of the order of the deconvolution denoted by N , in the case of the simplified Bardina's model [1, 10, 4], which is based on the approximation

$$(1.4) \quad \mathbb{S}(\mathbf{u}, \mathbf{u}) \approx \mathbb{S}(\bar{\mathbf{u}}, \bar{\mathbf{u}}) = \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} - \overline{\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}}.$$

The approximate deconvolution model (ADM in what follows) is deduced from the simplified Bardina's model by changing approximation (1.4) in

$$(1.5) \quad \mathbb{S}(\mathbf{u}, \mathbf{u}) \approx \mathbb{S}_N(\bar{\mathbf{u}}, \bar{\mathbf{u}}) = \bar{\mathbf{u}} \otimes \bar{\mathbf{u}} - \overline{D_N(\bar{\mathbf{u}}) \otimes D_N(\bar{\mathbf{u}})},$$

where the deconvolution operator D_N is such that

$$(1.6) \quad D_N = \sum_{n=0}^N (I - G)^n,$$

while still noting G the operator associated to the kernel G . We always have $\mathbb{S}_0(\bar{\mathbf{u}}, \bar{\mathbf{u}}) = \mathbb{S}(\bar{\mathbf{u}}, \bar{\mathbf{u}})$, and when $\|G\| < 1$ ¹ then for a fixed \mathbf{u} ,

$$(1.7) \quad \lim_{N \rightarrow \infty} \mathbb{S}_N(\bar{\mathbf{u}}, \bar{\mathbf{u}}) = \mathbb{S}(\mathbf{u}, \mathbf{u}).$$

Let $(\bar{\mathbf{u}}_N, \bar{p}_N)$ be the field calculated from approximation (1.5), that is the solution to the system

$$(1.8) \quad \begin{aligned} \partial_t \bar{\mathbf{u}}_N + \nabla \cdot (\overline{D_N(\bar{\mathbf{u}}_N) \otimes D_N(\bar{\mathbf{u}}_N)}) - \nu \Delta \bar{\mathbf{u}}_N + \nabla \bar{p}_N &= \bar{\mathbf{f}}, \\ \nabla \cdot \bar{\mathbf{u}}_N &= 0, \\ \bar{\mathbf{u}}_N(0, \mathbf{x}) &= \bar{\mathbf{u}}_0(\mathbf{x}), \end{aligned}$$

if any solution exists. Existence and uniqueness of a solution to System (1.8) was first proved in [6] when G is the usual Helmholtz filter in the 3D periodic case. More generally, if one can prove existence and uniqueness of a solution to system (1.8) for any G that satisfies

¹the operator norm is based on natural energy spaces the fields belongs to, which will be specified latter

(1.7), it is expected that the sequence $(\bar{\mathbf{u}}_N, \bar{p}_N)_{N \in \mathbb{N}}$ converges to $(\bar{\mathbf{u}}, \bar{p}) = (G\mathbf{u}, Gp)$, for some solution (\mathbf{u}, p) of the NSE.

Such convergence results has been proved in [2] in the 3D periodic case, when $G = G_{\alpha,p}$ is the generalised Helmholtz filter of order p with $p \geq 3/4$, where

$$(1.9) \quad G_{\alpha,p}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{T}_3^*} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{1 + \alpha^{2p} |\mathbf{k}|^{2p}},^2$$

after having proved existence and uniqueness of $(\bar{\mathbf{u}}_N, \bar{p}_N)$. In Definition (1.9), $\mathcal{T}_3 := 2\pi\mathbb{Z}^3/L$, $L > 0$ being the size of the computational box, and $\alpha > 0$ is the filter's width, usually of same magnitude of the mesh size in a numerical simulation (see [13] for further discussions).

This yields to consider the error modeling $\boldsymbol{\varepsilon}_N = \bar{\mathbf{u}} - \bar{\mathbf{u}}_N$, which goes to zero when N goes to infinity. It remains the issue of estimating the rate of convergence in terms of N . Staying within the 3D periodic framework and the generalised Helmholtz filter of order p ($p \geq 3/4$), we show in this paper that L^2 and H^1 norms of $\boldsymbol{\varepsilon}_N$ are of order $(p(N+1))^{-1/4p}$, (see our main result, Theorem 3.1 below).

To derive this rate of convergence, we first write the equation satisfied by $\boldsymbol{\varepsilon}_N$, by subtracting (1.8) to (1.2), which yields

$$(1.10) \quad \partial_t \boldsymbol{\varepsilon}_N + \nabla \cdot (\overline{D_N \boldsymbol{\varepsilon}_N \otimes D_N \bar{\mathbf{u}}_N}) - \nu \Delta \boldsymbol{\varepsilon}_N + \nabla r_N = -\nabla \cdot \bar{\boldsymbol{\tau}}_N - \nabla \cdot \overline{D_N \bar{\mathbf{u}} \otimes D_N \boldsymbol{\varepsilon}_N},$$

where $r_N = \bar{p} - \bar{p}_N$, and

$$(1.11) \quad \boldsymbol{\tau}_N = \mathbf{u} \otimes \mathbf{u} - D_N \bar{\mathbf{u}} \otimes D_N \bar{\mathbf{u}}$$

is the residual stress. By using successively an energy budget procedure and Gronwall's Lemma, we get an inequality satisfied by the norms of $A^{1/2} D_N \boldsymbol{\varepsilon}_N$ where $A = G^{-1}$ (in terms of operators), from which we deduce an inequality satisfied by the norms of $\boldsymbol{\varepsilon}_N$ itself (see Inequality (3.30) below). This inequality highlights the role played by the L^2 norm of the residual stress.

The weakness of this method is the regularity assumption that should be imposed on the field \mathbf{u} , which should be in $L^4(H^1)$. However, such proceedings are similar to usual uniqueness proofs about the NSE, always involving regularity assumptions.

It remains to estimate the L^2 norm of the residual stress (see Inequality (4.9)). We carry out this calculation by using Fourier series expansion and calculations outlined in Appendix 7, which, if they use elementary real analysis only, are not straightforward and were first speculated thanks to numerical and symbolic computations, before being rigorously proved.

We observe that the rate of convergence slows down as p increases in the range $[1, \infty[$. Moreover, the resulting bound goes to a constant that only depends on α and \mathbf{u} when p goes to infinity and N remains fixed. This is consistent with the idea that more large is p , then more smooth are the filtered fields, which should enlarge the SFS area. Therefore, one needs high orders of deconvolution to reconstruct well the resolved scale area for large values of p .

²In terms of operators $G_{\alpha,p} = (I - \alpha^{2p} \Delta^p)^{-1}$, where Δ^p denotes the p -Laplacien, and $A_{\alpha,p} = I - \alpha^{2p} \Delta^p = G^{-1}$

Then we consider the popular Gaussian filter,

$$(1.12) \quad \tilde{G}_\alpha(\mathbf{x}) = \left(\frac{6}{\alpha^2 \pi} \right)^{3/2} \exp \left(-\frac{6}{\alpha^2} \|\mathbf{x}\|^2 \right),$$

often used in LES. Applying the ADM theory for general abstract filters developed in [17], we deduce that the ADM is well-posed in the case of the Gaussian filter. Therefore, one may ask if there is convergence of the model to the filtered NSE when $N \rightarrow \infty$, and if yes what is the convergence rate.

The theory we develop for Helmholtz filters, does not apply to the Gaussian filter, because of a too strong convergence of its Fourier modes to zero as the wave number increases, although this is not an evidence that the convergence does not hold.

We argue by approximation in showing that the Gaussian filter can be approximated by

$$(1.13) \quad \tilde{G}_{\alpha,m}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{T}_3} \left(1 + \frac{\alpha^2 |\mathbf{k}|^2}{24m} \right)^{-m} e^{i\mathbf{k} \cdot \mathbf{x}},^3$$

when m goes to infinity. We show that our procedure is still valid for this sequence of filters, and we derive a bound of order $(N+1)^{-4m}$ from them. This bound goes to a constant depending on α and \mathbf{u} when m goes to infinity for a fixed N . Therefore, we cannot conclude that the deconvolution process converges to the filtered field $(\bar{\mathbf{u}}, \bar{p})$ in the case of the Gaussian filter. Because of the strong regularisation effect of this filter, we may conjecture that if such a convergence would hold, then it should be very low. Therefore, the deconvolution process seems to be not appropriate for the Gaussian filter. This remains an open issue.

The paper is organised as follows. We first fix the mathematical framework and recall the results of [2] useful for the continuation of the paper. We next detail how to bound the error modeling in terms of the residual stress, whose L^2 norm is then estimated by Fourier series expansions. We finally consider the Gaussian Filter by showing how to approximate it by the $\tilde{G}_{\alpha,m}$'s, the error modeling of which being then estimated. The paper finishes by a technical appendix including key results to derive estimates about the residual stress.

2 Mathematical framework

2.1 Space function

Throughout the paper, $\nu > 0$ and $\alpha > 0$ are fixed and we stay within the periodic case framework. The domain of study is the 3D torus

$$(2.1) \quad \mathbb{T}_3 = \mathbb{R}^3 / \mathcal{T}_3 \quad \text{where} \quad \mathcal{T}_3 := 2\pi \mathbb{Z}^3 / L,$$

for some given $L > 0$, which is the size of the computational box. All the fields we consider have zero mean on \mathbb{T}_3 . Let \mathbb{H}_s be the vector field space

$$(2.2) \quad \mathbb{H}_s = \left\{ \mathbf{w} = (w_1, w_2, w_3) = \sum_{\mathbf{k} \in \mathcal{T}_3^*} \hat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} : \sum_{\mathbf{k} \in \mathcal{T}_3^*} |\mathbf{k}|^{2s} |\hat{\mathbf{w}}_{\mathbf{k}}|^2 < \infty \right\},$$

³In terms of operators $\tilde{G}_{\alpha,m} = \left(1 - \frac{\alpha^2}{24m} \Delta \right)^{-m}$

equipped with the Hermitian structure defined by the inner product and its associated norm

$$(2.3) \quad (\mathbf{w}, \mathbf{v})_s = \sum_{\mathbf{k} \in \mathcal{T}_3^*} |\mathbf{k}|^{2s} \widehat{\mathbf{w}}_{\mathbf{k}} \cdot \widehat{\mathbf{v}}_{\mathbf{k}}^*, \quad \|\mathbf{w}\|_s = \left(\sum_{\mathbf{k} \in \mathcal{T}_3^*} |\mathbf{k}|^{2s} |\mathbf{w}_{\mathbf{k}}|^2 \right)^{\frac{1}{2}},$$

where

$$\forall \mathbf{k} = (k_1, k_2, k_3) \in \mathcal{T}_3, \quad |\mathbf{k}|^2 = k_1^2 + k_2^2 + k_3^2,$$

and z^* denotes the complex conjugate of z . It can be proved (see [14]) that for all $s \in \mathbb{R}$,

$$(2.4) \quad \mathbb{H}_s \text{ is isomorphic to } H^s(\mathbb{T}_3)^3, \quad (\mathbb{H}_s)' = \mathbb{H}_{-s},$$

and we denote

$$(2.5) \quad \forall (\mathbf{w}, \mathbf{v}) \in \mathbb{H}_{-s} \times \mathbb{H}_s, \quad {}_{-s}(\mathbf{w}, \mathbf{v})_s = \sum_{\mathbf{k} \in \mathcal{T}_3^*} \widehat{\mathbf{w}}_{\mathbf{k}} \cdot \widehat{\mathbf{v}}_{\mathbf{k}}^*$$

the duality pairing.

Let $\mathbf{H}_s \subset \mathbb{H}_s$ be the closed subspace of fields valued in \mathbb{R}^3 , characterized by

$$\mathbf{H}_s = \left\{ \mathbf{w} = \sum_{\mathbf{k} \in \mathcal{T}_3^*} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \in \mathbb{H}_s : \forall \mathbf{k} \in \mathcal{T}_3^*, \quad \widehat{\mathbf{w}}_{\mathbf{k}}^* = \widehat{\mathbf{w}}_{-\mathbf{k}} \text{ and } \mathbf{k} \cdot \widehat{\mathbf{w}}_{\mathbf{k}} = 0 \right\}.$$

On can show (see [14]) that

$$(2.6) \quad \mathbf{H}_s = \left\{ \mathbf{w} : \mathbb{T}_3 \rightarrow \mathbb{R}^3, \quad \mathbf{w} \in H^s(\mathbb{T}_3)^3, \quad \nabla \cdot \mathbf{w} = 0, \quad \int_{\mathbb{T}_3} \mathbf{w} \, d\mathbf{x} = \mathbf{0} \right\},$$

2.2 Operators

2.2.1 Kernel and filter

The general Helmholtz filter $\overline{\mathbf{w}} = G_{\alpha,p} \star \mathbf{w}$ is defined by the Fourier Series expansion of the kernel $G_{\alpha,p}$

$$(2.7) \quad G_{\alpha,p}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{T}_3^*} \widehat{G}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \widehat{G}_{\mathbf{k}} = \frac{1}{1 + \alpha^{2p} |\mathbf{k}|^{2p}}.$$

Viewed as an operator, one has $G_{\alpha,p} = (\mathbf{I} - \alpha^{2p} \Delta^{2p})^{-1}$. Furthermore, a given free divergence field \mathbf{w} being given, $\overline{\mathbf{w}}$ is solution of the PDE problem

$$(2.8) \quad \begin{aligned} -\alpha^{2p} \Delta^p \overline{\mathbf{w}} + \overline{\mathbf{w}} + \nabla r &= \mathbf{w} \quad \text{in } \mathbb{T}_3, \\ \nabla \cdot \overline{\mathbf{w}} &= 0 \quad \text{in } \mathbb{T}_3, \end{aligned}$$

where the Lagrange multiplier r is constant in this case.

From now, we write G instead of $G_{\alpha,p}$, and we denote in the same way kernel and operator. For all $s \geq 0$, G defines an isomorphism,

$$(2.9) \quad G : \left\{ \begin{array}{ll} \mathbb{H}_s & \longrightarrow \mathbb{H}_{s+2p} \\ \mathbf{w} = \sum_{\mathbf{k} \in \mathcal{T}_3^*} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} & \longrightarrow \overline{\mathbf{w}} = \sum_{\mathbf{k} \in \mathcal{T}_3^*} \widehat{G}_{\mathbf{k}} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \end{array} \right.,$$

and we set $A = G^{-1}$, characterised by its kernel

$$(2.10) \quad A(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{T}_3^*} \hat{A}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \hat{A}_{\mathbf{k}} = 1 + \alpha^{2p} |\mathbf{k}|^{2p}.$$

Notice that if $\mathbf{w} \in \mathbf{H}_s$, then $\bar{\mathbf{w}} \in \mathbf{H}_{s+2p}$ and the restriction of G to \mathbf{H}_s , still denoted by G is an isomorphism that maps \mathbf{H}_s onto \mathbf{H}_{s+2p} .

2.2.2 Deconvolution

Let D_N denote the deconvolution operator, characterised by the Kernel

$$D_N = \sum_{0 \leq n \leq N} (\mathbf{I} - G)^n = \sum_{\mathbf{k} \in \mathcal{T}_3} \hat{D}_{N,\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}},$$

where,

$$(2.11) \quad \hat{D}_{N,\mathbf{k}} = \sum_{n=0}^N \left(\frac{\alpha^{2p} |\mathbf{k}|^{2p}}{1 + \alpha^{2p} |\mathbf{k}|^{2p}} \right)^n = (1 + \alpha^{2p} |\mathbf{k}|^{2p}) \rho_{N,p,\mathbf{k}},$$

$$\rho_{N,p,\mathbf{k}} = 1 - \left(\frac{\alpha^{2p} |\mathbf{k}|^{2p}}{1 + \alpha^{2p} |\mathbf{k}|^{2p}} \right)^{N+1}.$$

The following holds [2]:

$$(2.12) \quad 1 \leq \hat{D}_{N,\mathbf{k}} \leq N + 1, \quad \forall \mathbf{k} \in \mathcal{T}_3,$$

$$(2.13) \quad \hat{D}_{N,\mathbf{k}} \approx (N + 1) \frac{1 + \alpha^{2p} |\mathbf{k}|^{2p}}{\alpha^{2p} |\mathbf{k}|^{2p}}, \quad \text{for large } |\mathbf{k}|,$$

$$(2.14) \quad \lim_{|\mathbf{k}| \rightarrow +\infty} \hat{D}_{N,\mathbf{k}} = N + 1,$$

$$(2.15) \quad \hat{D}_{N,\mathbf{k}} \leq (1 + \alpha^{2p} |\mathbf{k}|^{2p}) = \hat{A}_{\mathbf{k}}, \quad \forall \mathbf{k} \in \mathcal{T}_3,$$

where $\hat{A}_{\mathbf{k}}$ is defined by (2.10). We deduce from (2.12) and (2.14):

Lemma 2.1. *A real number $s \geq 0$ being given, the operator D_N is a isomorphism over \mathbb{H}_s , such that $1 \leq \|D_N\| \leq N + 1$. Moreover, the subspace of free divergence field \mathbf{H}_s is stable under the action of D_N . \square*

2.3 Former Results

This section aims to recall results of [2] about the system

$$(2.16) \quad \begin{aligned} \partial_t \bar{\mathbf{u}}_N + \nabla \cdot (\overline{D_N(\bar{\mathbf{u}}_N) \otimes D_N(\bar{\mathbf{u}}_N)}) - \nu \Delta \bar{\mathbf{u}}_N + \nabla \bar{p}_N &= \bar{\mathbf{f}}, \\ \nabla \cdot \bar{\mathbf{u}}_N &= 0, \\ \bar{\mathbf{u}}_N(0, \mathbf{x}) &= \bar{\mathbf{u}}_0(\mathbf{x}). \end{aligned}$$

Throughout the paper, we assume that \mathbf{u}_0 and \mathbf{f} satisfy,

$$(2.17) \quad \mathbf{u}_0 \in \mathbf{H}_0, \quad \mathbf{f} \in L^2([0, T] \times \mathbb{T}_3)^3,$$

and $\alpha > 0$ is fixed.

Definition 2.1 (Regular Weak solution). *We say that the couple $(\bar{\mathbf{u}}_N, \bar{p}_N)$ is a “regular weak solution” to system (2.16) if and only if the three following items are satisfied:*

1) REGULARITY

$$(2.18) \quad \bar{\mathbf{u}}_N \in L^2([0, T]; \mathbf{H}_{1+p}) \cap C([0, T]; \mathbf{H}_p),$$

$$(2.19) \quad \partial_t \bar{\mathbf{u}}_N \in L^2([0, T]; \mathbf{H}_0)$$

$$(2.20) \quad \bar{p}_N \in L^2([0, T]; H^1(\mathbb{T}_3)),$$

2) INITIAL DATA

$$(2.21) \quad \lim_{t \rightarrow 0} \|\bar{\mathbf{u}}_N(t, \cdot) - \bar{\mathbf{u}}_0\|_{\mathbf{H}_p} = 0,$$

3) WEAK FORMULATION

$$(2.22) \quad \forall \mathbf{v} \in L^2([0, T]; H^1(\mathbb{T}_3)^3),$$

$$(2.23) \quad \int_0^T \int_{\mathbb{T}_3} \partial_t \bar{\mathbf{u}}_N \cdot \mathbf{v} - \int_0^T \int_{\mathbb{T}_3} \overline{D_N(\bar{\mathbf{u}}_N) \otimes D_N(\bar{\mathbf{u}}_N)} : \nabla \mathbf{v} + \nu \int_0^T \int_{\mathbb{T}_3} \nabla \bar{\mathbf{u}}_N : \nabla \mathbf{v} \\ + \int_0^T \int_{\mathbb{T}_3} \nabla \bar{p}_N \cdot \mathbf{v} = \int_0^T \int_{\mathbb{T}_3} \bar{\mathbf{f}} \cdot \mathbf{v}.$$

□

Theorem 2.1. ([2]) *Assume $p \geq 3/4$. Problem (2.16) has a unique regular weak solution. Moreover, when $p \geq 1$,*

$$(2.24) \quad \partial_t \bar{\mathbf{u}}_N \in L^2([0, T], \mathbf{H}_{p-1}), \quad \bar{p}_N \in L^2([0, T], H^p(\mathbb{T}_3)).$$

□

Theorem 2.2. ([2]) *There exists a weak dissipative solution to the NSE (1.1)*

$$(\mathbf{u}, p) \in [L^2([0, T], \mathbf{H}_1) \cap L^2([0, T], \mathbf{H}_0)] \times L^{5/3}([0, T] \times \mathbb{T}_3)$$

such that from the sequence $(\bar{\mathbf{u}}_N, \bar{p}_N)_{N \in \mathbb{N}}$, one can extract a sub-sequence (still denoted $(\bar{\mathbf{u}}_N, \bar{p}_N)_{N \in \mathbb{N}}$) such that

$$(2.25) \quad \bar{\mathbf{u}}_N \rightarrow \bar{\mathbf{u}} \quad \begin{cases} \text{weakly in } L^2([0, T], \mathbf{H}_{1+p}(\mathbb{T}_3)^3) \cap L^\infty([0, T], \mathbf{H}_p), \\ \text{strongly in } L^r([0, T]; H^p(\mathbb{T}_3)^3), \quad \forall 1 \leq r < +\infty, \end{cases}$$

$$\bar{p}_N \rightarrow \bar{p} \quad \text{weakly in } L^2([0, T]; H^1(\mathbb{T}_3) \cap L^{5/3}([0, T]; W^{2p, 5/3}(\mathbb{T}_3)),$$

□

3 Estimate of the modeling error

3.1 Regularity assumption and main result

Let $(\bar{\mathbf{u}}_N, \bar{p}_N)$ be the solution of Problem (2.16). We assume that the limit $(\bar{\mathbf{u}}, \bar{p}) = (G\mathbf{u}, Gp)$ of $(\bar{\mathbf{u}}_N, \bar{p}_N)_{N \in \mathbb{N}}$ satisfies the regularity assumption

$$(3.1) \quad \mathbf{u} = A\bar{\mathbf{u}} \in L^4(\mathbf{H}_1).$$

By Sobolev injection Theorem, we deduce

$$(3.2) \quad \mathbf{u} \in L^4([0, T] \times \mathbb{T}_3).$$

Since (\mathbf{u}, p) is solution to the NSE, one has

$$(3.3) \quad \Delta p = -\nabla \cdot (\nabla \cdot (\mathbf{u} \otimes \mathbf{u})) + \nabla \cdot \mathbf{f},$$

which yields in the periodic case

$$(3.4) \quad p \in L^2([0, T] \times \Omega),$$

and we derive from the NSE,

$$(3.5) \quad \partial_t \mathbf{u} \in L^2([0, T], \mathbf{H}_{-1}).$$

Our main result is

Theorem 3.1. *Let $\varepsilon_N = \bar{\mathbf{u}} - \bar{\mathbf{u}}_N$ be the error modeling, and assume that (3.1) holds. Then we have*

$$(3.6) \quad \begin{aligned} & \|\varepsilon_N(t, \cdot)\|_0^2 + \alpha^{2p} \|\varepsilon_N(t, \cdot)\|_p^2 + \nu \int_0^t (\|\nabla \varepsilon_N(s, \cdot)\|_0^2 + \alpha^{2p} \|\nabla \varepsilon_N(s, \cdot)\|_p^2) ds \leq \\ & \frac{16C\alpha}{\nu(2p(N+1))^{1/2p}} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4 e^{\frac{1}{\nu^3} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4}. \end{aligned}$$

where C is a universal constant, as a product of Sobolev constants. ⁴

3.2 Modeling error and residual stress

Let ε_N and τ_N be the error modeling and the residual stress defined by

$$(3.7) \quad \begin{aligned} \varepsilon_N &= \bar{\mathbf{u}} - \bar{\mathbf{u}}_N, \\ \tau_N &= \mathbf{u} \otimes \mathbf{u} - D_N \bar{\mathbf{u}} \otimes D_N \bar{\mathbf{u}}. \end{aligned}$$

The equation satisfied by ε_N is derived by substracting (2.16) to the filtered NSE (1.2). Expressing the right hand side in terms of τ_N , we obtain

$$(3.8) \quad \partial_t \varepsilon_N + \nabla \cdot (\overline{D_N \varepsilon_N \otimes D_N \bar{\mathbf{u}}_N}) - \nu \Delta \varepsilon_N + \nabla r_N = -\nabla \cdot \bar{\tau}_N - \nabla \cdot (\overline{D_N \bar{\mathbf{u}} \otimes D_N \varepsilon_N}),$$

where $r_N = \bar{p} - \bar{p}_N$.

The aim of this section is to estimate ε_N in terms of τ_N . It addresses $A^{1/2} D_N^{1/2} \varepsilon_N$ rather than ε_N , since the natural multiplier to get an energy balance from equation (3.8) is $AD_N \varepsilon_N$, and formally $(\partial_t \varepsilon_N, AD_N \varepsilon_N) = (d/2dt) \|A^{1/2} D_N^{1/2} \varepsilon_N\|_0^2$. Once $A^{1/2} D_N^{1/2} \varepsilon_N$ is estimated, we derive bounds for ε_N (Corollary 3.1 below) by comparing the norms of the various operators we consider.

Theorem 3.2. *The following inequality holds:*

$$(3.9) \quad \begin{aligned} & \|A^{1/2} D_N^{1/2} \varepsilon_N(t, \cdot)\|_0^2 + \nu \int_0^t \|A^{1/2} D_N^{1/2} \varepsilon_N(s, \cdot)\|_1^2 ds \leq \\ & \frac{8}{\nu} e^{\frac{1}{\nu^3} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4} \int_0^t \|\tau_N(s, \cdot)\|_0^2 ds, \end{aligned}$$

for all $N > 0$ and $t \geq 0$. □

⁴For simplicity, we note $L^4(\mathbf{H}_1)$ instead of $L^4([0, T], \mathbf{H}_1)$

I found $4/\nu$ instead of $8/\nu$ and $27/\nu$ in the exponential instead of $1/\nu$: this needs to be checked

Proof. The proof is based on an energy equality satisfied by $A^{\frac{1}{2}}D_N^{\frac{1}{2}}\varepsilon_N$ to which one applies Gronwall's Lemma. To do so, we use $AD_N\varepsilon_N$ as multiplier in the (3.8) satisfied by ε_N and we integrate by parts.

The proof is divided into three steps. In a first one, we check that $AD_N\varepsilon_N$ is appropriate as multiplier to validate the procedure. In a second one, we perform integrations by parts. In a last step, we apply usual interpolation inequality to be in order to apply Gronwall's Lemma.

Step 3.i. *Consistency of the procedure.* We check the regularity of $A^{1/2}D_N^{1/2}\varepsilon_N$ and each factor in equation (3.8) one after each other, beginning with ε_N . The regularity assumption (3.1) combined with the regularization effect (2.9) of operator G , gives $\bar{\mathbf{u}} \in L^4([0, T], \mathbf{H}_{1+2p})$. Therefore, we have at least by (2.19) about $\bar{\mathbf{u}}_N$'s regularity,

$$(3.10) \quad \varepsilon_N \in L^2([0, T], \mathbf{H}_{1+p}) \subset L^2([0, T], \mathbb{H}_{1+p}),$$

where Applying Lemma 2.1 combined with (2.9), we get

$$(3.11) \quad AD_N\varepsilon_N \in L^2([0, T], \mathbb{H}_{1-p}).$$

We wish to prove now that each factor in equation (3.8) is at least in

$$L^2([0, T], \mathbb{H}_{p-1}) = (L^2([0, T], \mathbb{H}_{1-p}))'$$

(see subsection 2.1). To be synthetic, we write things as:

$$(3.12) \quad \left. \begin{array}{l} (2.19) \\ + \end{array} \right\} \begin{array}{l} (3.5) \\ (2.24) \end{array} \Rightarrow \left\{ \begin{array}{ll} \partial_t \varepsilon_N \in L^2([0, T], \mathbf{H}_0) & \text{if } 3/4 \leq p \leq 1, \\ \partial_t \varepsilon_N \in L^2([0, T], \mathbf{H}_{p-1}) & \text{if } p \geq 1. \end{array} \right.$$

When $3/4 \leq p \leq 1$, $\mathbf{H}_0 \hookrightarrow \mathbb{H}_0 \hookrightarrow \mathbb{H}_{p-1}$, and when $p \geq 1$, $\mathbf{H}_{p-1} \hookrightarrow \mathbb{H}_{p-1}$. In all cases,

$$(3.13) \quad \partial_t \varepsilon_N \in L^2([0, T], \mathbb{H}_{p-1}).$$

Similarly,

$$(3.14) \quad \left. \begin{array}{l} (2.20) \\ + \end{array} \right\} \begin{array}{l} (3.4) \\ (2.24) \end{array} \Rightarrow \left\{ \begin{array}{ll} \nabla r_N \in L^2([0, T] \times \mathbb{T}_3)^3 & \text{if } 3/4 \leq p \leq 1, \\ \nabla r_N \in L^2([0, T], H^{p-1}(\mathbb{T}_3)^3) & \text{if } p \geq 1, \end{array} \right.$$

that yields

$$(3.15) \quad \nabla r_N \in L^2([0, T], \mathbb{H}_{p-1}).$$

From the injection $\mathbf{H}_1 \hookrightarrow \mathbb{H}_1$, we deduce

$$(3.16) \quad (3.10) \Rightarrow \Delta \varepsilon_N \in L^2([0, T], \mathbb{H}_{p-1}).$$

Furthermore, as (\mathbf{u}, p) is a dissipative solution to the NSE, $\mathbf{u} \in L^\infty([0, T], \mathbb{H}_0)$, therefore $\bar{\mathbf{u}} \in L^\infty([0, T], \mathbb{H}_{2p})$, and by lemma 2.1, we get

$$(3.17) \quad D_N \bar{\mathbf{u}} \in L^\infty([0, T], \mathbb{H}_{2p}),$$

from which we conclude

$$(3.18) \quad \left. \begin{array}{l} (2.18) \\ + \end{array} \right\} \begin{array}{l} (3.17) \\ \text{lemma 2.1} \end{array} \Rightarrow D_N \varepsilon_N \in L^\infty([0, T], \mathbb{H}_p).$$

Since $p \geq 3/4$, we deduce from Sobolev injection Theorem $\mathbf{H}_p \hookrightarrow L^4(\mathbb{T}_3)^3$, that yields

$$(3.19) \quad \left. \begin{array}{l} (3.17) \\ + \\ (2.9) \end{array} \right\} \Rightarrow \nabla \cdot (\overline{D_N \epsilon_N \otimes D_N \bar{\mathbf{u}}_N}) \in L^\infty([0, T], \mathbb{H}_{2p-1}).$$

Similarly,

$$(3.20) \quad \nabla \cdot (\overline{D_N \bar{\mathbf{u}} \otimes D_N \epsilon_N}) \in L^\infty([0, T], \mathbb{H}_{2p-1}).$$

Finally, $\mathbf{u} \in L^\infty([0, T], \mathbb{H}_0)$ combined with (3.2) and properties of G and D_N already mentioned, yields

$$(3.21) \quad \nabla \cdot \bar{\tau}_N \in L^2([0, T], \mathbb{H}_{2p-1}).$$

Bringing together all these results, we conclude that when

$$\mathbb{A}_N = \partial_t \epsilon_N + \nabla \cdot (\overline{D_N \epsilon_N \otimes D_N \bar{\mathbf{u}}_N}) - \nu \Delta \epsilon_N + \nabla r_N + \nabla \cdot \bar{\tau}_N + \nabla \cdot (\overline{D_N \bar{\mathbf{u}} \otimes D_N \epsilon_N})$$

then $\mathbb{A}_N \in L^2([0, T], \mathbb{H}_{p-1})$. Therefore, the duality pairing ${}_{p-1}(\mathbb{A}_N, AD_N \epsilon_N)_{1-p}$, which makes consistent the multiplication of equation (3.8) by $AD_N \epsilon_N$. In what follows, we omit the subscripts when writing duality pairings.

Step 3.ii. *Energy equality.* Since all the operators we consider are self adjoint, the following holds (see [15]):

$$(3.22) \quad \begin{aligned} (\partial_t \epsilon_N, AD_N \epsilon_N) &= \frac{d}{2dt} \|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \epsilon_N\|_0^2, \\ (-\Delta \epsilon_N, AD_N \epsilon_N) &= \|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \epsilon_N\|_1^2. \end{aligned}$$

Furthermore, since $AD_N \epsilon_N$ has zero divergence, $(\nabla r_N, AD_N \epsilon_N) = 0$. Finally, as the operators commute with the differential operators,

$$(3.23) \quad \begin{aligned} (\nabla \cdot (\overline{D_N \epsilon_N \otimes D_N \mathbf{w}_N}), AD_N \epsilon_N) &= (A^{-1} \nabla \cdot (D_N \epsilon_N \otimes D_N \mathbf{w}_N), AD_N \epsilon_N) = \\ (A^{-1} \nabla \cdot (D_N \epsilon_N \otimes D_N \mathbf{w}_N), AD_N \epsilon_N) &= (\nabla \cdot (D_N \epsilon_N \otimes D_N \mathbf{w}_N), D_N \epsilon_N) = \\ &= ((D_N \mathbf{w}_N \cdot \nabla) D_N \epsilon_N, D_N \epsilon_N) = 0, \end{aligned}$$

because $D_N \mathbf{w}_N$ has zero divergence. Finally, arguing as in (3.23) to eliminate the bar in the integrals of right hand side, we get

$$(3.24) \quad \frac{d}{2dt} \|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \epsilon_N\|_0^2 + \nu \|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \epsilon_N\|_1^2 = (\tau_N, \nabla D_N \epsilon_N) - ((D_N \epsilon_N \cdot \nabla) D_N \bar{\mathbf{u}}, D_N \epsilon_N)$$

Step 3.iii. *Bounds and Gronwal's lemma.* We bound each term of the right hand side of (3.24) after each other. From Cauchy-Schwarz inequality combined with Young inequality, we get

$$(3.25) \quad |(\tau_N, \nabla D_N \epsilon_N)| \leq \frac{4}{\nu} \|\tau\|_0^2 + \frac{\nu}{4} \|D_N \epsilon_N\|_1^2.$$

In the same way, by using Ladyzenskaya's inequality [20] we obtain

$$(3.26) \quad \begin{aligned} |((D_N \epsilon_N \cdot \nabla) D_N \bar{\mathbf{u}}, D_N \epsilon_N)| &\leq \|D_N \epsilon_N\|_{L^4}^2 \|D_N \bar{\mathbf{u}}\|_1 \leq \\ &\|D_N \epsilon_N\|_0^{\frac{1}{2}} \|D_N \epsilon_N\|_1^{\frac{3}{2}} \|D_N \bar{\mathbf{u}}\|_1. \end{aligned}$$

The symbol of $D_N G$ is equal to $\rho_{N,p,\mathbf{k}} \in [0, 1]$ (see (2.11)). Therefore, we have $\|D_N \bar{\mathbf{u}}\|_1 \leq \|\mathbf{u}\|_1$. By Young inequality combined with (3.26), we obtain

$$(3.27) \quad |((D_N \boldsymbol{\varepsilon}_N \cdot \nabla) D_N \bar{\mathbf{u}}, D_N \boldsymbol{\varepsilon}_N)| \leq \frac{1}{\nu^3} \|\mathbf{u}\|_1^4 \|D_N \boldsymbol{\varepsilon}_N\|_0^2 + \frac{\nu}{4} \|D_N \boldsymbol{\varepsilon}_N\|_1^2.$$

We deduce from (2.15) that the symbol of D_N is less than the symbol of $A^{1/2} D_N^{1/2}$, which leads to

$$(3.28) \quad \|D_N \boldsymbol{\varepsilon}_N\|_0 \leq \|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \boldsymbol{\varepsilon}_N\|_0,$$

regardless of N . Combining (3.24), (3.25), (3.26) and (3.28) yields

$$(3.29) \quad \frac{d}{dt} \|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \boldsymbol{\varepsilon}_N\|_0^2 + \nu \|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \boldsymbol{\varepsilon}_N\|_1^2 \leq \frac{8}{\nu} \|\boldsymbol{\tau}\|_0^2 + \frac{1}{\nu^3} \|\mathbf{u}\|_1^4 \|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \boldsymbol{\varepsilon}_N\|_0^2$$

Inequality (3.9) results from inequality (3.29) thanks to a standard generalisation of Gronwall's lemma [8]. \square

Corollary 3.1. *The error modeling $\boldsymbol{\varepsilon}_N$ satisfies*

$$(3.30) \quad \|\boldsymbol{\varepsilon}_N(t, \cdot)\|_0^2 + \alpha^{2p} \|\boldsymbol{\varepsilon}_N(t, \cdot)\|_p^2 + \nu \int_0^t (\|\nabla \boldsymbol{\varepsilon}_N(s, \cdot)\|_0^2 + \alpha^{2p} \|\nabla \boldsymbol{\varepsilon}_N(s, \cdot)\|_p^2) ds \leq \frac{8}{\nu} e^{\frac{1}{\nu^3} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4} \int_0^t \|\boldsymbol{\tau}_N(s, \cdot)\|_0^2 ds,$$

for all $N > 0$ and $t \geq 0$. \square

Proof. Let $\mathbf{v} = \sum_{\mathbf{k} \in \mathcal{T}_3} \hat{\mathbf{v}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \in \mathbb{H}_p$. We observe that

$$(3.31) \quad \|A^{\frac{1}{2}} \mathbf{v}\|_0^2 = \sum_{\mathbf{k} \in \mathcal{T}_3} (1 + \alpha^{2p} |\mathbf{k}|^{2p}) |\hat{\mathbf{v}}_{\mathbf{k}}|^2 = \|\mathbf{v}\|_0^2 + \alpha^{2p} \|\mathbf{v}\|_p^2.$$

We first take $\mathbf{v} = D_N^{1/2} \boldsymbol{\varepsilon}_N$ in (3.31). By using (2.12), which yields the general formal inequality $\|\mathbf{w}\|_s \leq \|D_N^{1/2} \mathbf{w}\|_s$, we deduce the further inequality

$$(3.32) \quad \|\boldsymbol{\varepsilon}_N\|_0^2 + \alpha^{2p} \|\boldsymbol{\varepsilon}_N\|_p^2 \leq \|A^{1/2} D_N^{1/2} \boldsymbol{\varepsilon}_N\|_0^2.$$

We next take $\mathbf{v} = \partial_i D_N^{1/2} \boldsymbol{\varepsilon}_N$ in (3.31), which yields

$$(3.33) \quad \|\nabla \boldsymbol{\varepsilon}_N\|_0^2 + \alpha^{2p} \|\nabla \boldsymbol{\varepsilon}_N\|_p^2 \leq \|A^{1/2} D_N^{1/2} \boldsymbol{\varepsilon}_N\|_1^2.$$

We deduce (3.30) from (3.9) thanks to (3.32) and (3.33). \square

4 Residual stress and rate of convergence

Now that we have shown that the modeling error $\boldsymbol{\varepsilon}_N$ is driven by the L^2 norm of the residual stress $\boldsymbol{\tau}_N$, involving the $L^4(\mathbf{H}_1)$ norm of \mathbf{u} , it remains estimate the L^2 norm of $\boldsymbol{\tau}_N$, which what we aim to carry out in this section. Framework, assumptions and notations are those of section 3.

In what follows, S_s denotes the Sobolev constant⁵ in the injection $\mathbb{H}_s \hookrightarrow L^{s^*}(\mathbb{T}_3)^3$. To begin with, we show

⁵The constants S_1 and $S_{1/2}$ do not depend on L . One can prove that $S_1 \leq (16 + 3/\pi)^{1/3}$, see [14]. Unfortunately, we do not know any numerical bound for $S_{1/2}$, even such a bound may probably be found in the litterature

I found here
 $\frac{4}{\nu} \|\boldsymbol{\tau}\|_0^2 +$
 $\frac{27}{\nu^3} \|\mathbf{u}\|_1^4 \|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \boldsymbol{\varepsilon}_N\|_0^2$
: TO BE
CHECKED
AGAIN

Lemma 4.1. *The following inequalities holds true*

$$(4.1) \quad \|\tau_N\|_0^2 \leq 2C \|\mathbf{u}(t, \cdot)\|_1^2 \|\mathbf{u} - D_N \bar{\mathbf{u}}\|_{1/2}^2,$$

$$(4.2) \quad \|\mathbf{u} - D_N \bar{\mathbf{u}}\|_{1/2}^2 \leq \frac{\alpha}{(2p(N+1))^{1/2p}} \|\mathbf{u}\|_1^2,$$

where $C = S_1 S_{1/2}$.⁶ □

Proof. Step 4.i. *Proof of (4.1).* We write τ_N as

$$(4.3) \quad \tau_N = (\mathbf{u} - D_N \bar{\mathbf{u}}) \otimes \mathbf{u} + D_N \bar{\mathbf{u}} \otimes (\mathbf{u} - D_N \bar{\mathbf{u}}).$$

Therefore, combining Hölder inequality with $1/3 + 1/6 = 1/2$ for conjugation, to the Sobolev inequality $\|\mathbf{w}\|_{L^6} \leq S_1 \|\mathbf{w}\|_1$, we get

$$(4.4) \quad \|\tau\|_0^2 \leq 2S_1 \|\mathbf{u}\|_1^2 \|\mathbf{u} - D_N \bar{\mathbf{u}}\|_{L^3(\mathbb{T}_3)^3}^2,$$

To estimate $\|\mathbf{u} - D_N \bar{\mathbf{u}}\|_{L^3(\mathbb{T}_3)^3}^2$, we use the injection of $\mathbb{H}_{1/2}$ onto $L^3(\mathbb{T}_3)^3$ to obtain

$$(4.5) \quad \|\mathbf{u} - D_N \bar{\mathbf{u}}\|_{L^3(\mathbb{T}_3)^3}^2 \leq S_{1/2} \|\mathbf{u} - D_N \bar{\mathbf{u}}\|_{1/2}^2,$$

hence (4.1) by combining (4.4) and (4.5). □

Step 4.ii. *Proof of (4.2).* We deduce from (2.11),

$$(4.6) \quad \|\mathbf{u} - D_N \bar{\mathbf{u}}\|_{1/2}^2 = \sum_{\mathbf{k} \in \mathcal{T}_3} \left(\frac{\alpha^{2p} |\mathbf{k}|^{2p}}{1 + \alpha^{2p} |\mathbf{k}|^{2p}} \right)^{2(N+1)} |\mathbf{k}| |\hat{\mathbf{u}}_{\mathbf{k}}|^2,$$

We apply the technical inequality (7.6) proved in Appendix 7 below, with $x = \alpha^p |\mathbf{k}|^p$, $a = 2p(N+1) > 1$, $b = 0$, which yields

$$(4.7) \quad \left(\frac{\alpha^{2p} |\mathbf{k}|^{2p}}{1 + \alpha^{2p} |\mathbf{k}|^{2p}} \right)^{2p(N+1)} \leq \frac{\alpha^p |\mathbf{k}|^p}{\sqrt{2p(N+1)}}.$$

We raise both sides of (4.7) to the power $1/p$, we multiply the result by $|\mathbf{k}| |\hat{\mathbf{u}}_{\mathbf{k}}|^2$ and get

$$(4.8) \quad \left(\frac{\alpha^{2p} |\mathbf{k}|^{2p}}{1 + \alpha^{2p} |\mathbf{k}|^{2p}} \right)^{2(N+1)} |\mathbf{k}| |\hat{\mathbf{u}}_{\mathbf{k}}|^2 \leq \frac{\alpha}{(2p(N+1))^{1/2p}} |\mathbf{k}|^2 |\hat{\mathbf{u}}_{\mathbf{k}}|^2,$$

hence (4.2) from (4.6).

Corollary 4.1. *The following estimate holds*

$$(4.9) \quad \|\tau_N(t, \cdot)\|_0^2 \leq \frac{2C\alpha}{(2p(N+1))^{1/2p}} \|\mathbf{u}(t, \cdot)\|_1^4,$$

for all $t \in [0, T]$.

Inequality (4.9) results from (4.2) combined to (4.1). □

Summarizing: (3.30) + (4.9) \Rightarrow

$$(4.10) \quad \begin{aligned} & \|\varepsilon_N(t, \cdot)\|_0^2 + \alpha^{2p} \|\varepsilon_N(t, \cdot)\|_p^2 + \nu \int_0^t (\|\nabla \varepsilon_N(s, \cdot)\|_0^2 + \alpha^{2p} \|\nabla \varepsilon_N(s, \cdot)\|_p^2) ds \leq \\ & \frac{16C\alpha}{\nu(2p(N+1))^{1/2p}} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4 e^{\frac{1}{\nu^3} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4}. \end{aligned}$$

for all $N > 0$ and $t \geq 0$.

⁶Inequalities (4.1) and (4.2) both hold at any fixed time $t \in [0, T]$, which is not indicated here to reduce the notations.

5 Case of Gaussian filter

5.1 Framework

The Gaussian filter is specified by its kernel,

$$(5.1) \quad \tilde{G}_\alpha(\mathbf{x}) = \tilde{G}(\mathbf{x}) = \left(\frac{6}{\alpha^2 \pi} \right)^{3/2} \exp \left(-\frac{6}{\alpha^2} \|\mathbf{x}\|^2 \right),$$

where we omit the subscript α for simplicity. It can be shown that [18],

$$(5.2) \quad \tilde{G}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{T}_3} \tilde{G}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \quad \text{where} \quad \tilde{G}_{\mathbf{k}} = e^{-\frac{\alpha^2 |\mathbf{k}|^2}{24}}$$

Let $s \geq 0$ and $q \geq s$. There exists a constant C be such that

$$(5.3) \quad \forall \mathbf{k} \in \mathcal{T}_3^*, \quad \tilde{G}_{\mathbf{k}} |\mathbf{k}|^q \leq C |\mathbf{k}|^s.$$

Therefore,

$$(5.4) \quad \forall s \geq 0, \quad \forall \mathbf{u} \in \mathbb{H}_s, \quad \forall q \geq s, \quad \tilde{G}\mathbf{u} \in \mathbb{H}_q.$$

Let \mathbf{u} being given such that $\forall \mathbf{k} \in \mathcal{T}_3^*, |\hat{\mathbf{u}}_{\mathbf{k}}| = |\mathbf{k}|^{-1-q} \neq 0$ ($q \geq 0$). Such a vector field \mathbf{u} belongs to \mathbb{H}_q , but it easy checked that $\tilde{G}^{-1} \notin \mathbb{H}_s$ for any s . This is why the theory above about Helmholtz filters fails, since it is based on the fact that G defines an isomorphism between \mathbb{H}_s spaces.

However, ADM may be considered for the Gaussian filter, and the resulting model yields a well posed problem [17]. Moreover, we shall show in what follows that it can be approached in some sense, by a sequence of operators which fall within the framework of the theory exposed above.

5.2 Approximation of the Gaussian filter

we note that for all $\mathbf{k} \in \mathcal{T}_3^*$ fixed,

$$(5.5) \quad \tilde{G}_{\mathbf{k}} = \lim_{m \rightarrow \infty} \tilde{G}_{m,\mathbf{k}}, \quad \text{where} \quad \tilde{G}_{m,\mathbf{k}} = \left(1 + \frac{\alpha^2 |\mathbf{k}|^2}{24m} \right)^{-m}$$

Let \tilde{G}_m denotes the kernel

$$(5.6) \quad \tilde{G}_m(\mathbf{x}) = \sum_{\mathbf{k} \in \mathcal{T}_3} \left(1 + \frac{\alpha^2 |\mathbf{k}|^2}{24m} \right)^{-m} e^{i\mathbf{k} \cdot \mathbf{x}},$$

which corresponds to the operator, still denoted by \tilde{G}_m ,

$$(5.7) \quad \tilde{G}_m = \left(1 - \frac{\alpha^2}{24m} \Delta \right)^{-m}.$$

In a sense that needs to be precised, the sequence $(\tilde{G}_m)_{m \in \mathbb{N}}$ converges to \tilde{G} . To be more specific,

Lemma 5.1. *For all $\mathbf{k} \in \mathcal{T}_3^*$,*

$$(5.8) \quad |\tilde{G}_{\mathbf{k}} - \tilde{G}_{m,\mathbf{k}}| \leq \frac{2}{m}.$$

Proof. We prove in Appendix 7 the technical inequality (7.7),

$$\forall x \geq 0, \quad \forall m \geq 1, \quad \left| \left(1 + \frac{x}{m}\right)^{-m} - e^{-x} \right| \leq \frac{2}{m}.$$

We deduce inequality (5.8) in replacing in this inequality x by $\frac{\alpha^2 |\mathbf{k}|^2}{24}$. \square

The following corollary is straightforward:

Corollary 5.1. *For all $\mathbf{u} \in \mathbb{H}_s$,*

$$(5.9) \quad \|\tilde{G}\mathbf{u} - \tilde{G}_m\mathbf{u}\|_s \leq \frac{2}{m} \|\mathbf{u}\|_s.$$

In other words, there is weak star convergence of the sequence of operators $(\tilde{G}_m)_{m \in \mathbb{N}}$ to the Gaussian filter \tilde{G} in \mathbb{H}_s ($s \geq 0$).

5.3 Powers of the second order filter

In what follows, we put for m fixed,

$$(5.10) \quad \mu^2 = \frac{\alpha^2}{24m}.$$

and we denote by H_m the m^{th} power of the second order Helmholtz operator

$$(5.11) \quad H_m = (\mathbf{I} - \mu^2 \Delta)^{-m}.$$

Estimating the error modeling that corresponds to H_m yields estimates for the error modeling that corresponds to G_m . The theory developed above about Helmholtz operators applies to operator H_m . Indeed, let

$$(5.12) \quad \hat{H}_{m,\mathbf{k}} = \frac{1}{(1 + \mu^2 |\mathbf{k}|^2)^m}$$

be the symbol of H_m . Using the scalar inequality $1 + x^m \leq (1 + x)^m \leq 2^{m-1}(1 + x^m)$ for positive x , we get

$$(5.13) \quad \frac{1}{2^{m-1}(1 + \mu^{2m} |\mathbf{k}|^{2m})} \leq \hat{H}_{m,\mathbf{k}} \leq \frac{1}{1 + \mu^{2m} |\mathbf{k}|^{2m}}.$$

Using results of [2] (section 6), we deduce from (5.13) that the ADM corresponding to H_m has a unique regular weak solution $(\bar{\mathbf{u}}_N, \bar{p}_N)$, in the meaning of Definition 2.1 with $p = m$. Furthermore, this sequence of solution converges to some $(\bar{\mathbf{u}}, \bar{p})$ solution of the filtered NSE when N goes to infinity. Thus we can perform the programme to estimate ε_N in this case. We next prove.

⁷This estimate is uniform in \mathbf{k} , but unfortunately we cannot conclude from this the normal convergence of the kernel sequence $(\tilde{G}_m)_{m \in \mathbb{N}}$ because the serie $1/m$ is not convergent

Theorem 5.1. Let $\varepsilon_N = \bar{\mathbf{u}} - \bar{\mathbf{u}}_N$ be the error modeling corresponding to H_m , and assume that (3.1) still holds. Then we have ⁸

$$(5.14) \quad \|\varepsilon_N(t, \cdot)\|_0^2 + \mu^{2m} \|\varepsilon_N(t, \cdot)\|_m^2 + \nu \int_0^t (\|\nabla \varepsilon_N(s, \cdot)\|_0^2 + \mu^{2m} \|\nabla \varepsilon_N(s, \cdot)\|_m^2) ds \leq \frac{14C\mu m^{1/2}}{\nu(4(N+1))^{1/2m}} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4 e^{\frac{1}{\nu^3} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4}.$$

□

Proof. Thanks to (5.13), one can copy line by line proofs of Theorem (3.2) and Corollary (3.1) and derive

$$(5.15) \quad \|\varepsilon_N(t, \cdot)\|_0^2 + \mu^{2m} \|\varepsilon_N(t, \cdot)\|_m^2 + \nu \int_0^t (\|\nabla \varepsilon_N(s, \cdot)\|_0^2 + \mu^{2m} \|\nabla \varepsilon_N(s, \cdot)\|_m^2) ds \leq \frac{8}{\nu} e^{\frac{1}{\nu^3} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4} \int_0^t \|\tau_N(s, \cdot)\|_0^2 ds.$$

It remains to estimate $\|\tau_N(s, \cdot)\|_0^2$. Step 4.i in the proof of Lemma 4.1 can be recycled, so that (4.5) still holds in this case. Therefore, we only have to bound

$$(5.16) \quad \|\mathbf{u} - D_N \bar{\mathbf{u}}\|_{1/2}^2 = \sum_{\mathbf{k} \in \mathcal{T}_3^*} \left(1 - \frac{1}{(1 + \mu^2 |\mathbf{k}|^2)^m}\right)^{2(N+1)} |\mathbf{k}| |\hat{\mathbf{u}}_{\mathbf{k}}|^2,$$

where as usual $\mathbf{u} = \sum_{\mathbf{k} \in \mathcal{T}_3^*} \hat{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$. We apply the technical inequality (7.2) proved in Appendix 7 below, with $x = \mu |\mathbf{k}|$, $a = 2(N+1) > 1$, $m \geq 1$. We obtain

$$(5.17) \quad \left(1 - \frac{1}{(1 + \mu^2 |\mathbf{k}|^2)^m}\right)^{2(N+1)} \leq \frac{\sqrt{m} \mu}{(4(N+1))^{1/2m}} |\mathbf{k}|.$$

We multiply the result by $|\mathbf{k}| |\hat{\mathbf{u}}_{\mathbf{k}}|^2$ and get

$$(5.18) \quad \left(1 - \frac{1}{(1 + \mu^2 |\mathbf{k}|^2)^m}\right)^{2N+2} |\mathbf{k}| |\hat{\mathbf{u}}_{\mathbf{k}}|^2 \leq \frac{\sqrt{m} \mu}{(4(N+1))^{1/2m}} |\mathbf{k}|^2 |\hat{\mathbf{u}}_{\mathbf{k}}|^2,$$

hence

$$(5.19) \quad \|\mathbf{u} - D_N \bar{\mathbf{u}}\|_{1/2}^2 \leq \frac{\sqrt{m} \mu}{(4(N+1))^{1/2m}} \|\mathbf{u}\|_1,$$

which yields by (4.5),

$$(5.20) \quad \|\tau\|_0^2 \leq \frac{2\sqrt{m} \mu}{(4(N+1))^{1/2m}} \|\mathbf{u}\|_1^4,$$

giving (5.14) thanks to (5.15). □

⁸The constant C below is as in Theorem 3.1, inequality (3.9), see also Lemma 4.1.

5.4 Passing to the limit

From the results of subsection 5.3, we deduce thanks to the relation (5.10) that the ADM associated to the filter specified by (5.3), has a unique solution $(\bar{\mathbf{u}}_{N,m}, \bar{p}_{N,m})$ which converges to some solution $(\bar{\mathbf{u}}_m, \bar{p}_m)$, of the filtered NSE, by assuming that (\mathbf{u}_m, p_m) satisfies the regularity assumption (3.1).

Let $\varepsilon_{N,m} = \bar{\mathbf{u}}_m - \bar{\mathbf{u}}_{N,m}$ denotes the corresponding error modeling. Thanks to (5.14), we obtain⁹

$$\begin{aligned}
& \|\varepsilon_{N,m}(t, \cdot)\|_0^2 + \alpha^{2m}(24m)^{-m} \|\varepsilon_{N,m}(t, \cdot)\|_m^2 & + \\
& \nu \int_0^t (\|\nabla \varepsilon_{N,m}(s, \cdot)\|_0^2 & + \\
(5.21) \quad & \alpha^{2m}(24m)^{-m} \|\nabla \varepsilon_{N,m}(s, \cdot)\|_m^2) ds & \leq \\
& \frac{70C\alpha}{\nu(4(N+1))^{1/2m}} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4 e^{\frac{1}{\nu^3} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4}.
\end{aligned}$$

Without any convergence result about ADM's associated to Gaussian filter (5.1) when N goes to infinity, we cannot consider the corresponding error modeling, and therefore take the limit in (5.21) when m goes to infinity. Nevertheless, we observe that for a fixed N , the r.h.s of (5.1) converges, as $m \rightarrow \infty$, to some $C = C(\nu, \alpha, \mathbf{u}, C)$, which do not depend on N . We only can deduce a bound about the sup limit of the terms in the r.h.s.

6 Conclusions and open problems

6.1 Typical size of the constants

The main estimate (3.6) we get in the paper yields the rate of convergence to zero of the order modeling in the case of Helmholtz filter of order p . The bound involves a constant of the form

$$(6.1) \quad \kappa = \frac{1}{\nu} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4 e^{\frac{1}{\nu^3} \|\mathbf{u}\|_{L^4(\mathbf{H}_1)}^4}.$$

The number of iteration N required to reduce substantially the SFS area is driven by the size of the constant κ .

This constant involves gradients of the true velocity of the fluid, which may be huge. For instance, in some turbulent boundary layer, one may observe flows for which $\nabla \mathbf{u}$ is of order $3 \cdot 10^4 \text{ s}^{-1}$ in layers thick of about $10^{-1} m$. For such a air layer at 50° (that can be considered as incompressible) of width and length equal to $1 m$, over a time range of $1 s$, with $\nu \approx 20 \cdot 10^{-6} \text{ m}^2 \text{ s}^{-1}$, we find

$$\kappa \approx 10^{10^{28}} \text{ m}^4 \text{ s}^{-2},$$

which is a very huge constant. Therefore, even if the resolution would be of order $\alpha = 10^{-18} m$, to fully solve such a flow, the number of iteration N required to substantially reduce ε_N is so large that the deconvolution algorithm seems not suitable for practical simulations, which is in contradiction with results of [19], suggesting that very few iterations are sufficient to significantly reduce the SFS area.

⁹For the simplicity, we use $\mu m^{1/2} \leq 5\alpha$ instead of (5.10)

The rate of convergence as $(p(N+1))^{-1/4p}$ comes from estimating norms of the residual stress τ_N involved in the equation for ε_N , whereas the constant κ considered above comes from Gronwall's Lemma, which is known to lead to non optimal results. This yields the conjecture that the rate of convergence we found is optimal, which is not the case of the constant, that might be substantially improved. Furthermore, how the regularity assumption $\mathbf{u} \in L^4(\mathbf{H}_1)$ could be prevented ?

6.2 Gaussian Filter

It also remains the issue of convergence of ADM in the case of Gaussian filter. We conjecture that the convergence holds, but in a very weak sense, according to Corollary 5.1, a weak sense as yet undefined.

7 Appendix

This technical appendix aims at proving a general inequality that has been used in the proof of the estimate (4.9). The result is the following.

Theorem 7.1. *The scalar inequality*

$$(7.1) \quad \left(1 - \frac{1}{(1+x)^m}\right)^a \leq \frac{mx}{\sqrt[m]{a}}$$

holds true for any $x \geq 0$, $a, m \geq 1$. □

We consider the LHS function

$$h(x) = \left(1 - \frac{1}{(1+x)^m}\right)^a$$

and fixed parameters $a, m \geq 1$.

Its derivative is

$$h'(x) = am \left(1 - \frac{1}{(1+x)^m}\right)^{a-1} \frac{1}{(x+1)^{m+1}}$$

We apply to $h(x)$ the Lagrange intermediate formula on $[0, x]$ and get

$$\left(1 - \frac{1}{(1+x)^m}\right)^a = (x-0)h'(\psi) = xam \left(1 - \frac{1}{(1+\psi)^m}\right)^{a-1} \frac{1}{(\psi+1)^{m+1}}$$

for some $\psi \in (0, x)$.

The inequality becomes

$$xam \left(1 - \frac{1}{(1+\psi)^m}\right)^{a-1} \frac{1}{(\psi+1)^{m+1}} \leq \frac{mx}{\sqrt[m]{a}}$$

i.e.(after reducing xm from both sides)

$$a \left(1 - \frac{1}{(1+\psi)^m}\right)^{a-1} \frac{1}{(\psi+1)^{m+1}} \leq \frac{1}{\sqrt[m]{a}}$$

So now it's enough to prove that

$$a \left(1 - \frac{1}{(1+x)^m}\right)^{a-1} \frac{1}{(x+1)^{m+1}} \leq \frac{1}{\sqrt[m]{a}}$$

for any $x \geq 0$, $a, m \geq 1$.

To easy computations we make the substitution

$$y = \frac{1}{1+x} \in (0, 1)$$

and the inequality becomes

$$a (1 - y^m)^{a-1} y^{m+1} \leq \frac{1}{\sqrt[m]{a}}$$

or, after putting a on the RHS

$$(1 - y^m)^{a-1} y^{m+1} \leq a^{-1-1/m}$$

for any $y \in (0, 1)$, $a, m \geq 1$.

We denote the LSH above by $g(y) = (1 - y^m)^{a-1} y^{m+1}$

Its derivative with respect to y is

$$g'(y) = -y^m (1 - y^m)^{a-2} ((am + 1)y^m - (m + 1))$$

We see that the derivative vanishes at

$$y_0 = \left(\frac{m+1}{am+1}\right)^{1/m}$$

and is first positive on $[0, \left(\frac{m+1}{am+1}\right)^{1/m}]$ then negative on $[\left(\frac{m+1}{am+1}\right)^{1/m}, 1]$ therefore the maximum of g is attained at $\left(\frac{m+1}{am+1}\right)^{1/m}$ and is equal to

$$g\left(\left(\frac{m+1}{am+1}\right)^{1/m}\right) = \left(1 - \frac{m+1}{am+1}\right)^{a-1} \left(\frac{m+1}{am+1}\right)^{\frac{m+1}{m}} = \left(\frac{am-m}{am+1}\right)^{a-1} \left(\frac{m+1}{am+1}\right)^{\frac{m+1}{m}}$$

So now we need to show that

$$\left(\frac{am-m}{am+1}\right)^{a-1} \left(\frac{m+1}{am+1}\right)^{\frac{m+1}{m}} \leq a^{-1-1/m}$$

for $a, m \geq 1$.

Now polish a bit the formula above. In the first term on LHS we simplify m ,

$$\left(\frac{a-1}{a+1/m}\right)^{a-1} \left(\frac{m+1}{am+1}\right)^{\frac{m+1}{m}} \leq a^{-1-1/m}$$

In the bottom of the second term on LHS we pull out a ,

$$\left(\frac{a-1}{a+1/m}\right)^{a-1} a^{-1-1/m} \left(\frac{m+1}{m+1/a}\right)^{\frac{m+1}{m}} \leq a^{-1-1/m}$$

then cancel $a^{-1-1/m}$

$$\left(\frac{a-1}{a+1/m}\right)^{a-1} \left(\frac{m+1}{m+1/a}\right)^{\frac{m+1}{m}} \leq 1$$

Then in the second term on LHS we simplify m,

$$\left(\frac{a-1}{a+1/m}\right)^{a-1} \left(\frac{1+1/m}{1+1/(ma)}\right)^{\frac{m+1}{m}} \leq 1$$

Now let $z = 1/m \in (0, 1]$

The inequality becomes

$$\left(\frac{a-1}{a+z}\right)^{a-1} \left(\frac{1+z}{1+z/a}\right)^{1+z} \leq 1$$

for any $a \geq 1, z \in (0, 1]$.

We apply the natural log to both sides. We need to show that

$$(a-1)\ln(a-1) - (a-1)\ln(a+z) + (1+z)\ln(1+z) - (1+z)\ln(1+z/a) \leq 0$$

for any $a \geq 1, z \in (0, 1]$.

Let

$$f(a) = (a-1)\ln(a-1) - (a-1)\ln(a+z) + (1+z)\ln(1+z) - (1+z)\ln(1+z/a)$$

be the LHS in the inequality above as a function of a.

The derivative of f (with respect to a)

$$f'(a) = \ln(a-1) - \ln(z+a) + \frac{1+z}{a}$$

The second derivative is

$$f''(a) = -\frac{(z+1)(az-z-a)}{a^2(a-1)(z+a)}$$

Obviously, since $a \geq 1, z \in [0, 1)$ we have that $az - a \leq 0$, so $az - z - a \leq 0$ therefore

$$f''(a) = -\frac{(z+1)(az-z-a)}{a^2(a-1)(z+a)} \geq 0$$

We conclude that the first derivative is increasing, therefore

$$f'(a) \leq \lim_{a \rightarrow \infty} f'(a) = \lim_{a \rightarrow \infty} \left(\ln(a-1) - \ln(z+a) + \frac{1+z}{a} \right) = \lim_{a \rightarrow \infty} \left(\ln\left(\frac{a-1}{z+a}\right) + \frac{1+z}{a} \right) = 0$$

Therefore f' is negative, so f is decreasing. It follows that

$$f(a) \leq \lim_{a \rightarrow 1} f(a) = \lim_{a \rightarrow 1} \left((a-1)\ln(a-1) - (a-1)\ln(a+z) + (1+z)\ln(1+z) - (1+z)\ln(1+\frac{z}{a}) \right)$$

We know that in general

$$\lim_{x \rightarrow 0} x \ln(x) = 0$$

therefore, the above limit is zero

$$\begin{aligned} \lim_{a \rightarrow 1} (a-1) \ln(a-1) - (a-1) \ln(a+z) + (1+z) \ln(1+z) - (1+z) \ln(1+\frac{z}{a}) &= \\ &= 0 - 0 + (1+z) \ln(1+z) - (1+z) \ln(1+z) = 0 \end{aligned}$$

We conclude that $f(a) \leq 0$ which proves the inequality. \square

Corollary 7.1. *The scalar inequality*

$$(7.2) \quad \left(1 - \frac{1}{(1+x^2)^m}\right)^a \leq \frac{\sqrt{m}x}{\sqrt[2m]{2a}}$$

holds true for any $x \geq 0, a, m \geq 1$.

In the previous inequality we replace x with x^2 and get

$$(7.3) \quad \left(1 - \frac{1}{(1+x^2)^m}\right)^a \leq \frac{mx^2}{\sqrt[m]{a}}$$

for any $x \geq 0, a, m \geq 1$.

Replace in this inequality a with $2a$, still keep $a \geq 1$ (but works for $a \geq 1/2$)

$$(7.4) \quad \left(1 - \frac{1}{(1+x^2)^m}\right)^{2a} \leq \frac{mx^2}{\sqrt[m]{2a}}$$

for any $x \geq 0, a, m \geq 1$.

Now extract the square root of both sides

$$(7.5) \quad \left(1 - \frac{1}{(1+x^2)^m}\right)^a \leq \frac{\sqrt{m}x}{\sqrt[2m]{2a}}$$

Remark 7.1. *Setting $m = 1$ in the previous inequality gives*

$$(7.6) \quad \left(\frac{x^2}{1+x^2}\right)^a \leq \frac{x}{\sqrt{2a}} \leq \frac{x}{\sqrt{a}}$$

for any $x \geq 0, a \geq 1$.

The following inequality will be used to approximate the Gaussian filter with a power of the second order Helmholtz filter and calculate the accuracy of this approximation.

Theorem 7.2. *The scalar inequality*

$$(7.7) \quad |(1+x/n)^{-n} - e^{-x}| \leq \frac{2}{n}$$

is valid for any real $x \geq 0$ and any integer $n \geq 1$.

It is well-known that as a function of n (and fixed $x \geq 0$) the expression

$$(1 + x/n)^{-n}$$

is decreasing and converges to e^{-x} as $n \rightarrow \infty$. (this is elementary calculus, i omitted the proof.)

Therefore, the left hand side in (7.7) can be written as

$$|(1 + x/n)^{-n} - e^{-x}| = (1 + x/n)^{-n} - e^{-x} = e^{-n \ln(1+x/n)} - e^{-x} = e^{-n \ln(1+y)} - e^{-ny}$$

where $y = x/n \geq 0$.

Applying the intermediate value theorem of Lagrange (corresponding to the function $\xi \rightarrow e^{-n\xi}$) to the last term above we get that

$$e^{-n \ln(1+y)} - e^{-ny} = ne^{-n\xi}(y - \ln(1+y))$$

for some $\xi \in [\ln(1+y), y]$. Here we used $\ln(1+y) \leq y$ for $y \geq 0$. Since $e^{-n\xi} \leq e^{-n \ln(1+y)}$ we further have that

$$(7.8) \quad e^{-n \ln(1+y)} - e^{-ny} \leq ne^{-n \ln(1+y)}(y - \ln(1+y)) = n(1+y)^{-n}(y - \ln(1+y))$$

for any real $y \geq 0$ and integer $n \geq 1$

The term $y - \ln(1+y)$ appearing in the last term in the inequality above is estimated as

$$0 \leq y - \ln(1+y) \leq \frac{y^2}{2}$$

for any real $y \geq 0$

Going back to inequality (7.8) we finally have

$$e^{-n \ln(1+y)} - e^{-ny} \leq n(1+y)^{-n} \frac{y^2}{2}$$

We replace $y = x/n$ and obtain

$$\left(1 + \frac{x}{n}\right)^{-n} - e^{-x} \leq n \left(1 + \frac{x}{n}\right)^{-n} \frac{x^2}{2n^2} = x^2 \left(1 + \frac{x}{n}\right)^{-n} \frac{1}{2n}$$

But, as pointed out before, for any fixed x the function $n \rightarrow (1 + x/n)^{-n}$ is decreasing, so we have that for $n \geq 2$

$$\left(1 + \frac{x}{n}\right)^{-n} \leq \left(1 + \frac{x}{2}\right)^{-2} \leq \frac{4}{1+x^2}$$

Therefore, for $n \geq 2$

$$\left(1 + \frac{x}{n}\right)^{-n} - e^{-x} \leq x^2 \left(1 + \frac{x}{n}\right)^{-n} \frac{1}{2n} \leq \frac{4x^2}{1+x^2} \frac{1}{2n} \leq \frac{2}{n}$$

For $n = 1$ the left hand side of (7.7) becomes

$$(1+x)^{-1} - e^{-x} \leq (1+x)^{-1} + e^{-x} \leq 2$$

for any $x \geq 0$, so the inequality (7.7) is valid for $n = 1$ too.

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